Review exercise 1

1

$$2\tanh x - 1 = 0$$
$$2\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1$$
$$2e^x - 2e^{-x} = e^x + e^{-x}$$
$$e^x = 3e^{-x}$$
$$e^{2x} = 3$$
$$2x = \ln 3$$
$$x = \tfrac{1}{2}\ln 3$$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

You multiply both sides of this equation by e^x.

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

2

$$5 = 3\cosh x$$
$$5 = 3\left(\frac{e^x + e^{-x}}{2}\right)$$
$$10 = 3e^x + 3e^{-x}$$
$$3e^x - 10 + 3e^{-x} = 0$$
$$3e^{2x} - 10e^x + 3 = 0$$
$$(3e^x - 1)(e^x - 3) = 0$$
$$e^x = \tfrac{1}{3},\ e^x = 3$$
$$x = \ln\left(\tfrac{1}{3}\right),\ \ln 3$$

The wording of the question requires you to use the definition $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorise, $3y^2 - 10y + 3 = (3y - 1)(y - 3) = 0$. This gives $y = \frac{1}{3}$ and $y = 3$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 3$. With practice, the substitution can be omitted.

The answer $x = -\ln 3$ would also be acceptable.

3 The curves intersect when

$$5\sinh x = 4\cosh x$$

$$5\left(\frac{e^x - e^{-x}}{2}\right) = 4\left(\frac{e^x + e^{-x}}{2}\right)$$

$$5e^x - 5e^{-x} = 4e^x + 4e^{-x}$$

$$e^x = 9e^{-x}$$

$$e^{2x} = 9$$

$$2x = \ln 9$$

$$x = \tfrac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Using the law of logarithms $n\ln a = \ln a^n$ with $n = \frac{1}{2}$ and $a = 9$.

$$y = 5\sinh(\ln 3) = 5\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) = \frac{5}{2}\times\left(3 - \frac{1}{3}\right)$$

$$= \frac{5}{2}\times\frac{8}{3} = \frac{20}{3}$$

$$p = 3,\ q = \frac{20}{3}$$

$e^{\ln 3} = 3$ and $e^{-\ln 3} = e^{\ln 1 - \ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$, using $\ln 1 = 0$ and the law of logarithms $\ln a - \ln b = \ln\frac{a}{b}$.

4

$$5\cosh x - 2\sinh x = 11$$

$$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$$

$$5e^x + 5e^{-x} - 2e^x + 2e^{-x} = 22$$

$$3e^x - 22 + 7e^{-x} = 0$$

$$3e^{2x} - 22e^x + 7 = 0$$

$$(3e^x - 1)(e^x - 7) = 0$$

$$e^x = \tfrac{1}{3},\ 7$$

$$x = \ln\tfrac{1}{3},\ \ln 7$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorising $3y^2 - 22y + 7 = (3y-1)(y-7) = 0$. This gives $y = \frac{1}{3}$ and $y = 7$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 7$. With practice, the substitution can be omitted.

5

$$6\sinh 2x + 9\cosh 2x = 7$$

$$6\left(\frac{e^{2x} - e^{-2x}}{2}\right) + 9\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 7$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ replacing x by $2x$.

$$6e^{2x} - 6e^{-2x} + 9e^{2x} + 9e^{-2x} = 14$$

$$15e^{2x} - 14 + 3e^{-2x} = 0$$

You multiply this equation throughout by e^{2x}.

$$15e^{4x} - 14e^{2x} + 3 = 0$$

$$(3e^{2x} - 1)(5e^{2x} - 3) = 0$$

$$e^{2x} = \frac{1}{3}, \frac{3}{5}$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

$$2x = \ln\frac{1}{3}, \ \ln\frac{3}{5}$$

$$x = \frac{1}{2}\ln\frac{1}{3}, \ \frac{1}{2}\ln\frac{3}{5}$$

$$p = \frac{1}{3}, \frac{3}{5}$$

6 a

$$\sinh x + 2\cosh x = k$$

$$\frac{e^x - e^{-x}}{2} + 2\left(\frac{e^x + e^{-x}}{2}\right) = k$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$e^x - e^{-x} + 2e^x + 2e^{-x} = 2k$$

$$3e^x - 2k + e^{-x} = 0$$

$$3e^{2x} - 2ke^x + 1 = 0$$

Let $y = e^x$

$$3y^2 - 2ky + 1 = 0$$

$$y = \frac{2k \pm \sqrt{(4k^2 - 12)}}{6}$$

Using the quadratic formula $y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

$$= \frac{k \pm \sqrt{(k^2 - 3)}}{3} \quad \textbf{(1)}$$

For real y

$k^2 - 3 \geqslant 0 \Rightarrow k \geqslant \sqrt{3}, \ k \leqslant -\sqrt{3}$

If $x \leqslant -\sqrt{3}$, then both $\frac{k + \sqrt{k^2 - 3}}{3}$ and $\frac{k - \sqrt{(k^2 - 3)}}{3}$ are negative.

As $y = e^x > 0$ for all real x, $k \leqslant -\sqrt{3}$ is rejected.

$k \geqslant \sqrt{3}$.

b Using **(1)** above with $k = 2$

You could solve the equation in part **b** without using part **a** but it is efficient to use the work you have already done.

$$y = e^x = \frac{2 \pm \sqrt{(4-3)}}{3} = \frac{2 \pm 1}{3}$$

$$e^x = 1, \tfrac{1}{3} \Rightarrow x = \ln 1, \ \ln\tfrac{1}{3} = 0, \ -\ln 3$$

7 a

$$\cosh^2 x - \sinh^2 x \equiv \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\equiv \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$$

$$\equiv \frac{4}{4} \equiv 1, \text{ as required.}$$

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Rewriting the equation in terms of the exponential definitions of the hyperbolic functions, it becomes:

$$\frac{1}{\frac{e^x - e^{-x}}{2}} - \frac{2}{\frac{e^x - e^{-x}}{e^x + e^{-x}}} = 2$$

$$\frac{2}{e^x - e^{-x}} - \frac{2\left(e^x + e^{-x}\right)}{e^x - e^{-x}} = 2$$

$$2 - 2e^x - 2e^{-x} = 2e^x - 2e^{-x}$$

$$4e^x = 2$$

$$x = \ln\tfrac{1}{2} = -\ln 2$$

8 a

$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= 2 \times \frac{e^{2x} + 2 + e^{-2x}}{4} - 1$$

$$= \frac{2e^{2x}}{4} + \frac{4}{4} + \frac{2e^{-2x}}{4} - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x, \text{ as required}$$

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Using the result in part **a**

$$\cosh 2x - 5\cosh x = 2$$

$$2\cosh^2 x - 1 - 5\cosh x = 2$$

$$2\cosh^2 x - 5\cosh x - 3 = 0$$

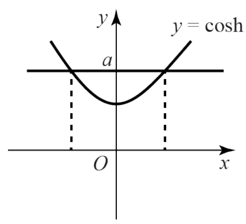
$$(2\cosh x + 1)(\cosh x - 3) = 0$$

$$\cosh x = -\tfrac{1}{2},\ \cosh x = 3$$

$\cosh x = -\frac{1}{2}$ is impossible

$$x = \pm\operatorname{arcosh} 3 = \pm\ln(3 + \sqrt{8})$$

If $\operatorname{arcosh} x > 0$ then $\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$.
However if $\cosh x = a$, where $a > 0$, then there are two answers $x = \pm\ln(a + \sqrt{(a^2 - 1)})$



These answers can also be written as $x = \ln(3 \pm \sqrt{8})$.

9 a

$$4\cosh^3 x - 3\cosh x = 4\left(\frac{e^x + e^{-x}}{2}\right)^3 - 3\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^x + 3e^x + 3e^{-x} + e^{-3x}}{2} - \frac{3e^x + 3e^{-x}}{2}$$

$$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x, \text{ as required.}$$

Using the binomial expansion
$(e^x + e^{-x})^3$
$= (e^x)^3 + 3(e^x)^2 \cdot e^{-x}$
$+ 3e^x(e^{-x})^2 + (e^{-x})^3$
$= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}$.

b $\cosh 3x = 5\cosh x$

Using the result in part **a**

$$4\cosh^3 x - 3\cosh x = 5\cosh x$$

$$4\cosh^3 x - 8\cosh x = 0$$

$$4\cosh x(\cosh^2 x - 2) = 0$$

As for all x, $\cosh x \geqslant 1$,

$$\cosh x = \sqrt{2}$$

$$x = \pm\ln\left(\sqrt{2}+1\right)$$

$$-\ln\left(\sqrt{2}+1\right) = \ln\left(\frac{1}{\sqrt{2}+1}\right) = \ln\left(\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \ln\left(\frac{\sqrt{2}-1}{1}\right) = \ln\left(\sqrt{2}-1\right)$$

The solutions of $\cosh 3x = 5\cosh x$, as natural logarithms, are $x = \ln\left(\sqrt{2} \pm 1\right)$.

There are 3 possible answers to this cubic, $\cosh x = 0$, $\cosh x = -\sqrt{2}$ and $\cosh x = \sqrt{2}$. As for all real x, $\cosh x \geqslant 1$ only the last of the three gives real values of x.

Using $\text{arcosh}\, x = \ln\left(x + \sqrt{(x^2 - 1)}\right)$

10 a

$$\cosh A \cosh B - \sinh A \sinh B$$

$$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \frac{1}{4}\left(e^{A+B} + e^{-A+B} + e^{A-B} + e^{-A-B} - e^{A+B} + e^{-A+B} + e^{A-B} - e^{-A-B}\right)$$

$$= \frac{1}{4}\left(2e^{-A+B} + 2e^{A-B}\right) = \frac{e^{A-B} + e^{-(A-B)}}{2}$$

$$= \cosh(A - B), \text{ as required.}$$

When multiplying out the brackets you must be careful to obtain all eight terms with the correct signs.

You use the definition $\cosh x = \dfrac{e^x + e^{-x}}{2}$ with $x = A + B$.

b

$$\cosh x \cosh 1 - \sinh x \sinh 1 = \sinh x$$

$$\cosh x \cosh 1 = \sinh x(1 + \sinh 1)$$

$$\tanh x = \frac{\cosh 1}{1 + \sinh 1}$$

You expand $\cosh(x-1)$ using the result of part **a**.

Divide both sides of this equation by $\cosh x(1 + \sinh 1)$ and use $\tanh x = \dfrac{\sinh x}{\cosh x}$.

$$\tanh x = \frac{\frac{e+e^{-1}}{2}}{1 + \frac{e-e^{-1}}{2}} = \frac{e + e^{-1}}{2 + e - e^{-1}} = \frac{e^2 + 1}{e^2 + 2e - 1}, \text{ as required.}$$

11 a Let $y = \text{arsinh}\, x$

Then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \dfrac{2x + \sqrt{(4x^2 + 4)}}{2}$

$= \dfrac{2x + 2\sqrt{(x^2 + 1)}}{2} = x + \sqrt{(x^2 + 1)}$

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2 + 1)}\right]$, as required.

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2 + 1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

b

$\text{arsinh}\,(\cot\theta) = \ln\left[\cot\theta + \sqrt{(1 + \cot^2\theta)}\right]$

$= \ln(\cot\theta + \text{cosec}\,\theta)$

$= \ln\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{1}{\sin\theta}\right) = \ln\left(\dfrac{\cos\theta + 1}{\sin\theta}\right)$

$= \ln\left(\dfrac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \ln\left(\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = \ln\left(\cot\dfrac{\theta}{2}\right)$, as required.

Using the result of part **a** with $x = \cot\theta$.

Using $\text{cosec}^2\theta = 1 + \cot^2\theta$.

You use both double angle formulae $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x \cos x$ with $2x = \theta$.

12 a Let $y = \text{artanh}\, x$

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y}$$

$$= \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$xe^{2y} + x = e^{2y} - 1$$

$$e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \frac{1+x}{1-x}$$

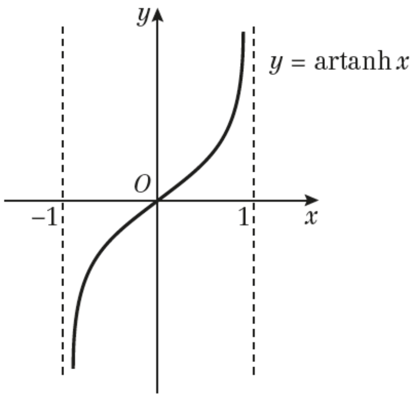
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } |x| < 1 \text{ as required.}$$

You have been asked to prove a standard result in this question. You should learn the proof of this and other similar results as part of your preparation for the examination.

Make e^{2y} the subject of the formulas and then take logarithms.

b



You need to be able to sketch the graphs of the hyperbolic and inverse hyperbolic functions. When you sketch a graph you should show any important features of the curve. In this case, you should show the asymptotes $x = -1$ and $x = 1$ of the curve.

c

$$x = \tanh\left[\ln\sqrt{(6x)}\right]$$

$$\ln\sqrt{(6x)} = \text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\ln\sqrt{(6x)} = \ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$\sqrt{(6x)} = \sqrt{\left(\frac{1+x}{1-x}\right)}$$

Squaring

$$6x = \frac{1+x}{1-x}$$

$$6x - 6x^2 = 1 + x$$

$$6x^2 - 5x + 1 = (3x-1)(2x-1) = 0$$

$$x = \tfrac{1}{2}, \tfrac{1}{3}$$

You use the result in part **a**.

As you have squared this equation, you might have introduced an incorrect solution. It would be sensible to check on your calculator that $x = \frac{1}{2}, \frac{1}{3}$ are solutions of $x = \tanh\left[\ln\sqrt{(6x)}\right]$. In this case, both are correct.

13 a

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)+\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$

$$=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$

$$=\ln\frac{1-(1-x^2)}{x^2}=\ln\frac{x^2}{x^2}=\ln 1=0$$

Hence

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\text{ as required.}$$

There are a number of different ways of starting this question. The method used here begins by using the log rule $\log a+\log b=\log ab$.

This is the difference of two squares $a^2-b^2=(a-b)(a+b)$ with $a=1$ and $b=\sqrt{(1-x^2)}$.

b Let $y=\operatorname{arcosh}\frac{1}{x}=\operatorname{arsech} x$

$$\operatorname{sech} y=x$$

$$\frac{2}{e^y+e^{-y}}=x$$

$$2=xe^y+xe^{-y}$$

$$xe^{2y}-2e^y+x=0$$

$$e^y=\frac{2\pm\sqrt{(4-4x^2)}}{2x}=\frac{1\pm\sqrt{(1-x^2)}}{x}$$

$$y=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\ \ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)$$

$$=\pm\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\text{ using the result of a}$$

$$y=\operatorname{arsech} x=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\text{ as required.}$$

Multiply throughout by e^y and treat the result as a quadratic in e^y.

Either of the two answers is possible but it is conventional to take $0<\operatorname{arsech} x\leqslant 1$.

c Using $\operatorname{sech}^2 x=1-\tanh^2 x$

$$3\tanh^2 x-4\operatorname{sech} x+1=0,$$

$$3-3\operatorname{sech}^2 x-4\operatorname{sech} x+1=0$$

$$3\operatorname{sech}^2 x+4\operatorname{sech} x-4=0$$

$$(3\operatorname{sech} x-2)(\operatorname{sech} x+2)=0$$

$$\operatorname{sech} x=\tfrac{2}{3}$$

$$x=\pm\ln\left(\frac{1+\sqrt{(1-\frac{4}{9})}}{\frac{2}{3}}\right)=\pm\ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$\operatorname{sech} x=-2$ is impossible with real values of x.

$x=\ln\left(\frac{3\pm\sqrt{5}}{2}\right)$ is another correct form of the answer.

14 a

$$\cosh 3\theta = \cosh(2\theta + \theta)$$
$$= \cosh 2\theta \cosh \theta + \sinh 2\theta \sinh \theta$$

Let $\cosh \theta = c$ and $\sinh \theta = s$

$$\cosh 3\theta = (2c^2 - 1)c + 2sc \times s$$
$$= 2c^3 - c + 2s^2 c$$
$$= 2c^3 - c + 2(c^2 - 1)c$$
$$= 2c^3 - c + 2c^2 - 2c$$
$$= 4\cosh^3\theta - 3\cosh\theta$$

$$\cosh 5\theta = \cosh(3\theta + 2\theta) = \cosh 3\theta \cosh 2\theta + \sinh 3\theta \sinh 2\theta$$

$$\cosh 3\theta \cosh 2\theta = (4c^3 - 3c)(2c^2 - 1)$$
$$= 8c^5 - 10c^3 + 3c$$

$$\sinh 3\theta \sinh 2\theta = \sin(2\theta + \theta)\sinh 2\theta$$
$$= (\sinh 2\theta \cosh \theta + \cosh 2\theta \sinh \theta)\sinh 2\theta$$
$$= (2sc \times c + (2c^2 - 1)s)2sc$$
$$= 2(4c^2 - 1)s^2 c$$
$$= 2(4c^2 - 1)(c^2 - 1)c$$
$$= 8c^5 - 10c^3 + 2c$$

Combining the results

$$\cosh 5\theta = 8c^2 - 10c^3 + 3c + 8c^5 - 10c^3 + 2c$$
$$= 16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta$$

In a complicated calculation like this, it is sensible to use the abbreviated notation suggested here but, if you intend to use a notation like this, you should state the notation in the solution so that the marker knows what you are doing.

You use the 'double angle' for hyperbolics $\cosh 2\theta = 2\cosh^2\theta - 1$ and $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$. The signs in these formulae can be worked out using Osborn's rule.

b $2\cosh 5x + 10\cosh 3x + 20\cosh x = 243,$

Letting $\cosh x = c$ and using the results in **a**

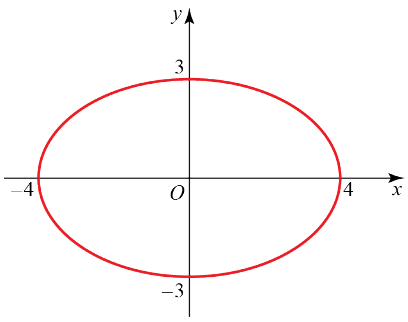
$$32c^5 - 40c^3 + 10c + 40c^3 - 30c + 20c = 243$$

$$c^5 = \frac{243}{32} \Rightarrow c = \frac{3}{2}$$

$$x = \pm\text{arcosh}\frac{3}{2} \approx \pm 0.96$$

You can use an inverse hyperbolic button on your calculator to find $\text{arcosh}\frac{3}{2}$.

15 a



When you draw a sketch, you should show the important features of the curve. When drawing an ellipse, you should show that it is a simple closed curve and indicate the coordinates of the points where the curve intersects the axes.

15 b $b^2 = a^2(1-e^2)$

$9 = 16(1-e^2) = 16 - 16e^2$

$e^2 = \frac{16-9}{16} = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

The formula you need for calculating the eccentricity and the coordinates of the foci are given in the Edexcel formula booklet you are allowed to use in the examination. You should be familiar with the formulae in that booklet. You should quote any formulae you use in your solution.

c The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{7}}{4}, 0\right) = \left(\pm\sqrt{7}, 0\right)$

16 a $b^2 = a^2(e^2-1)$

$4 = 16(e^2-1) = 16e^2 - 16$

$e^2 = \frac{16+4}{16} = \frac{20}{16} = \frac{5}{4}$

$e = \frac{\sqrt{5}}{2}$

The formula for calculating the eccentricity is $b^2 = a^2(e^2-1)$

It is important not to confuse this with the formula for calculating the eccentricity of an ellipse $b^2 = a^2(1-e^2)$

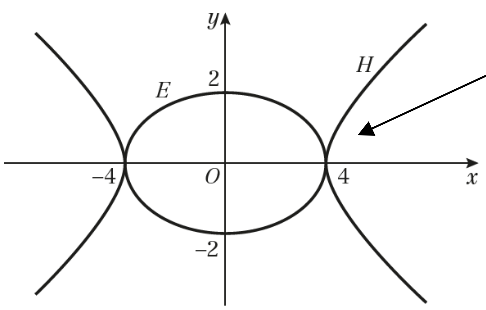
b The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{5}}{2}, 0\right) = \left(\pm 2\sqrt{5}, 0\right)$

The formulae for the foci of an ellipse and a hyperbola are the same $(\pm ae, 0)$

Therefore the distance between the foci is $2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$

c



this sketch, you should show where the curves cross the axes. Label which curve is H and which is E. These two curves touch each other on the x-axis.

17 a $b^2 = a^2\left(1-e^2\right)$

$4 = 9\left(1-e^2\right) = 9 - 9e^2$

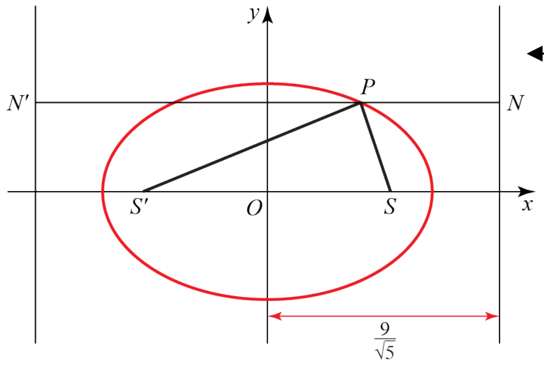
$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

As the coordinates of the foci of an ellipse are $(\pm ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2\left(1-e^2\right)$ with $a = 3$ and $b = 2$

The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = \left(\pm\sqrt{5}, 0\right)$

17 b



In this question, you are not asked to draw a diagram but with questions on coordinate geometry it is usually a good idea to sketch a diagram so you can see what is going on.

The equations of the directrices are $x = \pm\dfrac{a}{e}$

$$x = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$$

Let the line through P parallel to the x-axis intersect the directrices at N and N', as shown in the diagram

$$N'N = 2\times\frac{9}{\sqrt{5}} = \frac{18}{\sqrt{5}}$$

If you introduce points, like N and N' here, you should define them in your solution and mark them on your diagram. This helps the examiner follow your solution.

The focus directrix property of the ellipse gives that

$$SP = ePN \quad \text{and} \quad S'P = ePN'$$

$$\begin{aligned} SP + S'P &= ePN' + ePN' \\ &= e(PN + PN') = eN'N \\ &= \frac{\sqrt{5}}{3}\times\frac{18}{\sqrt{5}} = 6, \text{ as required.} \end{aligned}$$

18 a $3x^2 + 4y^2 = 12$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$b^2 = a^2\left(1-e^2\right)$$

$$3 = 4\left(1-e^2\right) = 4 - 4e^2$$

$$e^2 = \frac{4-3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$$

You divide this equation by 12

Comparing the result with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a^2 = 4$ and $b^2 = 3$ and you use $b^2 = a^2(1 - e^2)$ to calculate e.

18 b $3x^2+4y^2=12$

Differentiate implicitly with respect to x

$6x+8y\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{6x}{8y}=-\frac{3x}{4y}$

At $\left(1,\frac{3}{2}\right)$

$\frac{dy}{dx}=\frac{-3\times 1}{4\times\frac{3}{2}}=-\frac{1}{2}$

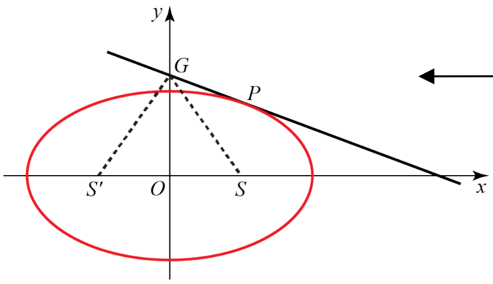
Differentiating implicitly using the chain rule, $\frac{d}{dx}\left(4y^2\right)=\frac{dy}{dx}\frac{d}{dy}\left(4y^2\right)=\frac{dy}{dx}\times 8y$

Using $y-y_1=m(x-x_1)$, an equation of the tangent is

$y-\frac{3}{2}=-\frac{1}{2}(x-1)=-\frac{1}{2}x+\frac{1}{2}$

$y=-\frac{1}{2}x+2$

c



Sketching a diagram makes it clear that the area of the triangle is to be found using the standard expression $\frac{1}{2}\text{base}\times\text{height}$ with the base $S'S$ and the height OG.

The coordinates of S are

$(ae,0)=\left(2\times\frac{1}{2},0\right)=(1,0)$

By symmetry, the coordinates of S' are $(-1,0)$

The y-coordinate of G is given by

$y=0+2=2$

You find the y-coordinate of G by substituting $x=0$ into the answer to part **a**.

$\Delta SS'G=\frac{1}{2}\text{base}\times\text{height}$

$=\frac{1}{2}S'S\times OG$

$=\frac{1}{2}2\times 2=2$

19 a S_2 has coordinates $\left(\frac{a}{2}\sqrt{3}, 0\right)$

Hence

$e = \frac{\sqrt{3}}{2}$

$b^2 = a^2\left(1 - e^2\right)$

$= a^2\left(1 - \frac{3}{4}\right) = \frac{a^2}{4}$ *

Comparing $\left(\frac{a}{2}\sqrt{3}, 0\right)$ with the formula for the focus $(ae, 0), e = \frac{\sqrt{3}}{2}$

An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

You are given that a is the semi-major axis, so a can be left in the equation. The data in the question does not include b, so b must be replaced.

Using *

$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{4}} = 1$

The required equation is

$\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$

$x^2 + 4y^2 = a^2$

b Equations of the directrices are

$x = \pm\frac{a}{e} = \pm\frac{a}{\frac{\sqrt{3}}{2}} = \pm\frac{2a}{\sqrt{3}}$

c From * above, $b = \frac{a}{2}$

19 d For Q

$$\left(a\cos\phi,\frac{1}{2}a\sin\varphi\right)=\left(a\cos\frac{\pi}{4},\frac{1}{2}a\sin\frac{\pi}{4}\right)$$
$$=\left(\frac{a}{\sqrt{2}},\frac{a}{2\sqrt{2}}\right)=\left(\frac{a\sqrt{2}}{2},\frac{a\sqrt{2}}{4}\right)$$

For P

$$\left(a\cos\phi,\frac{1}{2}a\sin\varphi\right)=\left(a\cos\frac{\pi}{2},\frac{1}{2}a\sin\frac{\pi}{2}\right)$$
$$=\left(0,\frac{a}{2}\right)$$

For PQ

$$\frac{y-\frac{a}{2}}{\frac{a\sqrt{2}}{4}-\frac{a}{2}}=\frac{x-0}{\frac{a\sqrt{2}}{2}-0}$$
$$\frac{4y-2a}{\sqrt{2}-2}=\frac{2x}{\sqrt{2}}$$
$$4\sqrt{2}y-2\sqrt{2}a=(2\sqrt{2}-4)x$$
$$\left(4-2\sqrt{2}\right)x+4\sqrt{2}y-2\sqrt{2}a=0$$

Using the formula from module C1 for a line, $\dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1}$

The a cancels throughout the denominators of this equation. On the left-hand side

$$\frac{4\left(y-\frac{a}{2}\right)}{4\left(\frac{\sqrt{2}}{4}-\frac{1}{2}\right)}=\frac{4y-2a}{\sqrt{2}-2}$$

Dividing throughout by $2\sqrt{2}$

$\left(\sqrt{2}-1\right)x+2y-a=0$, as required.

20 a

$$b^2=a^2(1-e^2)$$
$$4=9(1-e^2)=9-9e^2$$
$$e^2=\frac{9-4}{9}=\frac{5}{9}\Rightarrow e=\frac{\sqrt{5}}{3}$$

The formulae you need for calculating the eccentricity, the coordinates of the foci, and the equations of the directrices are given in the Edexcel formula booklet you are allowed to use in the examination. However, it wastes time checking your textbook every time you need to use these formulae and it is worthwhile remembering them. **Remember** to quote any formulae you use in your solution.

b The coordinates of the foci are

$$(\pm ae,0)=\left(\pm3\times\frac{\sqrt{5}}{3},0\right)=(\pm\sqrt{5},0)$$

The equation of the directrices are

$$x=\pm\frac{a}{e}=\pm\frac{3}{\frac{\sqrt{5}}{3}}=\pm\frac{9}{\sqrt{5}}$$

20 c

$x = 3\cos\theta,\ y = 2\sin\theta$

$\frac{dx}{d\theta} = -3\sin\theta, \frac{dy}{d\theta} = 2\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2\cos\theta}{3\sin\theta}$

$y - y_1 = m(x - x_1)$

$y - 2\sin\theta = \frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

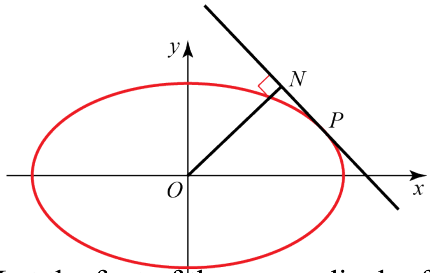
$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

$2x\cos\theta + 3y\sin\theta = 6(\cos^2\theta + \sin^2\theta) = 6$

Divide this line throughout by 6

$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$, as required.

20 d



Let the foot of the perpendicular from O to the tangent at P be N.
Using $mm' = -1$ the gradient of ON is given by

$$m' = -\frac{1}{\frac{dy}{dx}} = \frac{3\sin\theta}{2\cos\theta}$$

An equation of ON is $y = \dfrac{3\sin\theta}{3\sin\theta}$ $\quad *$

Eliminating y between equation $*$ and the answer to part c

$$\frac{x\cos\theta}{3} + \frac{\sin\theta}{2}\left(\frac{3\sin\theta}{2\cos\theta}x\right) = 1$$

$$x\left(\frac{4\cos^2\theta + 9\sin^2\theta}{12\cos\theta}x\right) = 1$$

$$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$x = \dfrac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$ and $y = \dfrac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$ are parametric equations of the locus. Eliminating θ between them to obtain a Cartesian equation is not easy and you will need to use the printed answer to help you.

Substituting this expression for x into equation $*$

$$y = \frac{3\sin\theta}{2\cos\theta} \times \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta} = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$$x^2 + y^2 = \left(\frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2 + \left(\frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2$$

$$= \frac{144\cos^2\theta + 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{36(4\cos^2\theta + 9\sin^2\theta)}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

$$9x^2 + 4y^2 = \frac{9\times 144\cos^2\theta + 4\times 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{1296\cos^2\theta + 1296\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{1296}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \left(\frac{36}{4\cos^2\theta + 9\sin^2\theta}\right)^2 = (x^2 + y^2)^2$$

The locus of N is $(x^2 + y^2)^2 = 9x^2 + 4y^2$, as required.

21 a

$x = a\cos\theta,\ y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = -\dfrac{b\cos\theta}{a\sin\theta}$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin\theta}{b\cos\theta}$$

$$y - y_1 = m'(x - x_1)$$

$$y - b\sin\theta = \frac{a\sin\theta}{b\sin\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

Divide this equation throughout by sin θ cos θ

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$, as required

b Substituting $y = 0$ in the result to part **a**

$$ax\sec\theta = a^2 - b^2$$

$$x = \frac{a^2 - b^2}{a}\cos\theta$$

You find the x-coordinate of G by substituting $y = 0$ into the equation of the normal at P and solving the resulting equation for x.

$P : (a\cos\theta,\ b\sin\theta),\ G : \left(\dfrac{a^2 - b^2}{a}\cos\theta,\ 0\right)$

The coordinates $(x_M,\ y_M)$ of M the midpoint of PG are given by

$\left(\dfrac{x_1 + x_2}{2},\ \dfrac{y_1 + y_2}{2}\right)$.

$$x_M = \frac{a\cos\theta + \frac{a^2 - b^2}{a}\cos\theta}{2}$$

$$= \frac{\cos\theta}{2}\left(\frac{a^2 + a^2 - b^2}{a}\right) = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta$$

$$y_M = \frac{b\sin\theta + 0}{2} = \frac{b\sin\theta}{2}$$

Hence, the coordinates of M are

$\left(\dfrac{2a^2 - b^2}{2a}\cos\theta,\ \dfrac{b}{2}\sin\theta\right)$, as required.

21 c For M

$$x=\left(\frac{2a^2-b^2}{2a}\right)\cos\theta,\ y=\left(\frac{b}{2}\right)\sin\theta$$

$$\cos\theta\frac{x}{\left(\frac{2a^2-b^2}{2a}\right)},\ \sin\theta=\frac{y}{\left(\frac{b}{2}\right)}$$

$$\cos^2\theta+\sin^2\theta=1$$

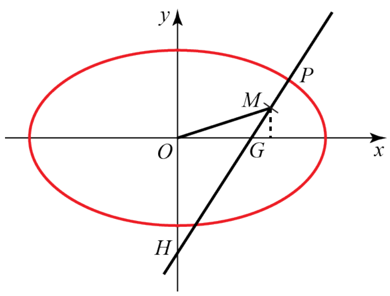
$$\frac{x^2}{\left(\frac{2a^2-b^2}{2a}\right)^2}+\frac{y^2}{\left(\frac{b}{2}\right)^2}=1$$

Any curve with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse. If you are asked to show that a locus is an ellipse, it is sufficient to show that it has a Cartesian equation of this form.

This is an ellipse. A Cartesian equation of this ellipse is

$$\frac{4a^2x^2}{\left(2a^2-b^2\right)^2}+\frac{4y^2}{b^2}=1$$

d



Substituting $x = 0$ into the equation of the normal

$$-by\operatorname{cosec}\theta=a^2-b^2\Rightarrow y=-\frac{a^2-b^2}{b}\sin\theta$$

Hence $OH=\frac{a^2-b^2}{b}\sin\theta$

$$\frac{\text{area }\Delta OMG}{\text{area }\Delta OGH}=\frac{y-\text{coordinate of }M}{OH}$$

$$=\frac{\left(\frac{b}{2}\right)\sin\theta}{\frac{a^2-b^2}{b}\sin\theta}$$

$$=\frac{b^2}{2(a^2-b^2)},\text{ as required.}$$

The triangles OMG and OGH can be looked at as having the same base OG.
As the area of a triangle is
$\frac{1}{2}\times\text{base}\times\text{height}$, triangles with the same base will have areas proportional to their heights. The height of the triangle OGM is shown by a dotted line in the diagram and is given by the y-coordinate of M.

22 In the first Cartesian quadrant, the ellipse is the graph of the function $y = 4\sqrt{1-\frac{x^2}{8^2}}$

The area it encloses for x greater than 4 is $4\int_4^8 \sqrt{1-\frac{x^2}{8^2}}\, \mathrm{d}x$; we solve the integral by putting $x = 8\sin u$, getting:

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2 u \; \mathrm{d}u =$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1+\cos 2u \; \mathrm{d}u =$$

$$= \frac{16\pi}{3} + 16\left[-\frac{\sqrt{3}}{4}\right] =$$

$$= \frac{16\pi}{3} - 4\sqrt{3}$$

In order to get the total area we need the area of the triangle PQO, where O is the origin and Q is the projection of P on the x-axis. The area of this triangle is easily $4\sqrt{3}$: therefore, the area of the shaded region is $\frac{16}{3}\pi$ and $a = \frac{16}{3}$

23 a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mc+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Multiply this equation throughout by a^2b^2
Then multiply out the brackets and collect the terms together as a quadratic in x.

As the line is a tangent this equation has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$\cancel{a^2m^2c^2} - \cancel{a^2m^2c^2} + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$c^2 = a^2m^2 + b^2, \text{ as required.}$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

23 b $(3,4) \in y = mx + c$

Hence $4 = 3m + c \Rightarrow c = 4 - 3m$ **(1)**

For this ellipse, $a = 4$ and $b = 5$ and the result in part **a** becomes

The tangents have equations of the form $y = mx + c$ and $x = 3$, $y = 4$ must satisfy this relation.

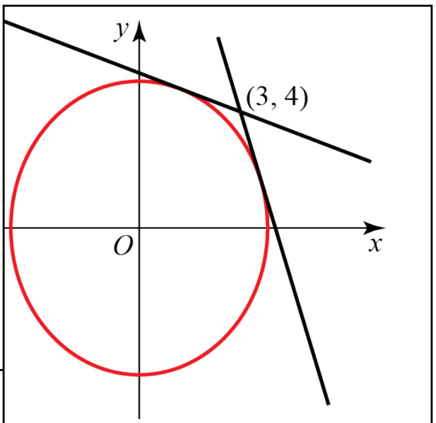
$c^2 = 16m^2 + 25$ **(2)**

Substituting **(1)** into **(2)**

$$(4-3m)^2 = 16m^2 + 25$$
$$16 - 24m + 9m^2 = 16m^2 + 25$$
$$7m^2 + 24m + 9 = (m+3)(7m+3) = 0$$
$$m = -3, -\frac{3}{7}$$

If $m = -3$, $c = 4–3\ m = 4 + 9 = 13$

If $m = -\frac{3}{7}$, $c = 4 - 3m = 4 + \frac{9}{7} = \frac{37}{7}$



There are two tangents to the ellipse which pass through (3, 4)
Both have negative gradients.

The equations of the tangents are

$$y = -3x + 13 \text{ and } y = -\frac{3}{7}x + \frac{37}{7}$$

24 a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$
$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$
$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$
$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0, \text{ as required.}$$

Multiply this equation throughout by a^2b^2
Then multiply out the brackets and collect the terms together as a quadratic in x.

b As the line is a tangent the result of part **a** has repeated roots '$b^2 - 4ac = 0$'

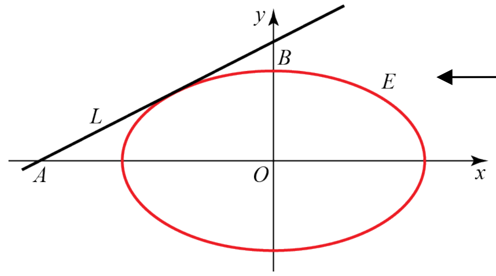
$$4a^4m^2c^2 - 4(b^2 + a^2m^2)a^2(c^2 - b^2) = 0$$

Divide this equation throughout by $4a^2$

$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$
$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$
$$c^2 = a^2m^2 + b^2, \text{ as required.}$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

24 c



As $c^2 = a^2m^2 + b^2$, $y = mx + c$ could have the forms
$y = mx \pm \sqrt{(b^2 + a^2m^2)}$
However, the question specifies that the tangent crosses the positive y-axis. As the line has a positive y intercept, you can reject the negative possibility.

An equation of L is $y = mx + \sqrt{b^2 + a^2m^2}$
For A $y = 0$

$$0 = mx + \sqrt{b^2 + a^2m^2} \Rightarrow x = -\frac{\sqrt{b^2 + a^2m^2}}{m}$$

Hence $OA = \dfrac{\sqrt{b^2 + a^2m^2}}{m}$

For B $x = 0$

$y = \sqrt{b^2 + a^2m^2}$

Hence $OB = \sqrt{b^2 + a^2m^2}$

The area of triangle OAB, T say, is given by

$$T = \frac{1}{2}OA \times OB = \frac{1}{2}\frac{\sqrt{b^2 + a^2m^2}}{m}\sqrt{b^2 + a^2m^2}$$
$$= \frac{b^2 + a^2m^2}{2m}$$

24 d $T = \dfrac{b^2 + a^2m^2}{2m} = \dfrac{1}{2}b^2m^{-1} + \dfrac{1}{2}a^2m$

For a minimum

$\dfrac{dT}{dm} = -\dfrac{1}{2}b^2m^{-2} + \dfrac{1}{2}a^2 = 0$

$\dfrac{b^2}{m^2} = a^2 \Rightarrow m^2 = \dfrac{b^2}{a^2}$

As L has a positive gradient

$m = \dfrac{b}{a}$

The diagram shows that the tangent has a positive gradient and so the possible value $-\dfrac{b}{a}$ can be ignored.

$\dfrac{d^2T}{dm^2} = b^2m^{-3} = \dfrac{b^2}{m^3}$

At $m = \dfrac{b}{a}, \dfrac{d^2T}{dm^2} = \dfrac{b^2}{m^3} = \dfrac{a^3}{b} > 0$ and so this gives a minimum value of

$T = \dfrac{b^2 + a^2\left(\dfrac{b}{a}\right)^2}{2\left(\dfrac{b}{a}\right)} = \dfrac{2b^2}{2\left(\dfrac{b}{a}\right)} = ab$, as required.

e At $m = \dfrac{b}{a}, c^2 = a^2m^2 + b^2 = a^2\left(\dfrac{b}{a}\right)^2 + b^2 = 2b^2$

Substituting $m = \dfrac{b}{a}$ and $c = \sqrt{2}b$ into the result in part **a**.

$\left(b^2 + a^2 \times \dfrac{b^2}{a^2}\right)x^2 + 2a^2 \times \dfrac{b}{a} \times \sqrt{2}bx + a^2(2b^2 - b^2) = 0$

$2b^2x^2 + 2ab^2\sqrt{2}x + a^2b^2 = 0$

Divide this equation throughout by b^2

$2x^2 + 2a\sqrt{2}x + a^2 = 0$

$\left(\sqrt{2}x + a\right)^2 = 0$

As the line is a tangent, this quadratic must factorise to a complete square. If you cannot see the factors, you can use the quadratic formula.

$x = -\dfrac{a}{\sqrt{2}}$

25 a To find an equation of the tangent at P.

$x = \cosh t,\ y = \sinh t$

$\frac{dx}{dt} = \sinh t,\quad \frac{dy}{dt} = \cosh t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cosh t}{\sinh t}$

Using $y - y_1 = m(x - x_1)$

$$y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t)$$

$$y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t$$

$$y \sinh t = x \cosh t - (\cosh^2 t - \sinh^2 t)$$

$$= x \cosh t - 1$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$

$x \cosh t - y \sinh t = 1$ **(1)**

To find the equation of the normal at P.

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{\sinh t}{\cosh t}$$

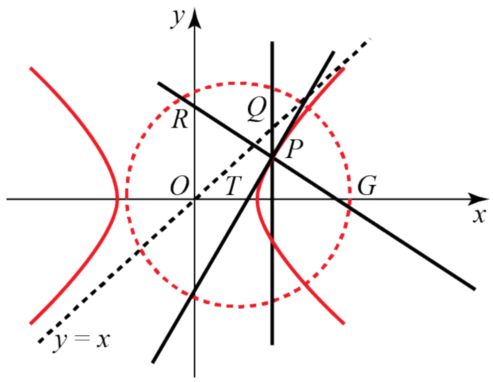
$$y - y_1 = m'(x - x_1)$$

$$y - \sinh t = -\frac{\sinh t}{\cosh t}(x - \cosh t)$$

$$y \cosh t - \sinh t \cosh t = -x \sinh t + \sinh t \cosh t$$

$x \sinh t + y \cosh t = 2 \sinh t \cosh t$ **(2)**

25 b



Substitute $y = 0$ into **(2)**

$x\sinh t = 2\sinh t\cosh t$

$x = 2\cosh t$

The coordinates of G are ($2\cosh t$, 0)

The x-coordinate of Q is $\cosh t$

The asymptote in the first quadrant has equation $y = x$

Hence the coordinates of Q are ($\cosh t$, $\cosh t$)

The gradient of GQ is given by $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{0 - \cosh t}{2\cosh t - \cosh t} = -1$

As the gradient of $y = x$ is 1 and $1 \times -1 = -1$, GQ is perpendicular to the asymptote.

To find the coordinates of G, you substitute $y = 0$ into the equation of the normal found in part **a**.

To asymptotes to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = \pm\dfrac{b}{a}x$

These formulae are given in the Edexcel formulae booklet. With the hyperbola $a = b = 1$ and the asymptotes are $y = \pm x$

The asymptote in the first quadrant has equation $y = x$

c Substitute $y = 0$ into **(1)**

$x\cosh t = 1 \Rightarrow x = \dfrac{1}{\cosh t}$

The coordinates of T are $\left(\dfrac{1}{\cosh t}, 0\right)$

Substitute $x = 0$ into **(2)**

$y\cosh t = 2\sinh t\cosh t \Rightarrow y = 2\sinh t$

The coordinates of R are $(0, 2\sinh t)$

$TG = 2\cosh t - \dfrac{1}{\cosh t}$

$$TR^2 = OR^2 + OT^2 = (2\sinh t)^2 + \left(\frac{1}{\cosh t}\right)^2$$

$$= 4\sinh^2 t + \frac{1}{\cosh^2 t} = 4(\cosh^2 t - 1) + \frac{1}{\cosh^2 t}$$

$$= 4\cosh^2 t - 4 + \frac{1}{\cosh^2 t}$$

$$= \left(2\cosh t - \frac{1}{\cosh t}\right)^2 = TG^2$$

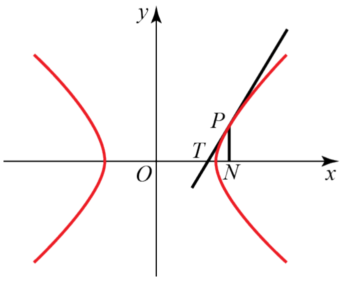
Hence $TR = TG$ and R lies on the circle with centre at T and radius TG.

To find the coordinates of T, you substitute $y = 0$ into the equation of the tangent found in part **a**.

To find the coordinates of R, you substitute $x = 0$ into the equation of the normal found in part **a**.

If a circle can be drawn through R with centre T and radius TG then TR must also be a radius of the circle. So you can solve the problem by showing that TR and TG have the same length.

26



To find the coordinates of T, it is easiest to carry out your calculation in terms of a parameter. As the question specifies no particular parametric form, you can choose your own. The hyperbolic form has been used here but (a sec t, b tan t) would work as well and there are other possible alternatives.

Let the point P have coordinates $(a\cosh t, b\sinh t)$

To find an equation of the tangent PT,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\sinh t, \frac{\mathrm{d}y}{\mathrm{d}t} = b\cosh t$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b\cosh t}{a\sinh t}$$

Using $y - y_1 = m(x - x_1)$

$$y - b\sinh t = \frac{b\cosh t}{a\sinh t}(x - a\cosh t)$$

To find the x-coordinate of T, you substitute $y = 0$ into an equation of the tangent at P, so first you must obtain an equation for the tangent.

$$ay\sinh t - ab\sinh^2 t = bx\cosh t - ab\cosh^2 t$$

$$ay\sinh t = bx\cosh t - ab(\cosh^2 t - \sinh^2 t)$$

$$= bx\cosh t - ab$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$

For T, $y = 0$

$$bx\cosh t = ab \Rightarrow x = \frac{a}{\cosh t}$$

The coordinates of N are (a cosh t, 0)

$$OT \cdot ON = \frac{a}{\cosh t} \times a\cosh t = a^2, \text{ as required.}$$

27 a $\frac{\mathrm{d}x}{\mathrm{d}t} = a\sec t\tan t, \frac{\mathrm{d}y}{\mathrm{d}t} = b\sec^2 t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t} = \frac{b}{a\sin t}$$

To find the gradient of the tangent, you use a version of the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Using $mm' = -1$, the gradient of the normal

is given by $m' = -\frac{a\sin t}{b}$

An equation of the normal is

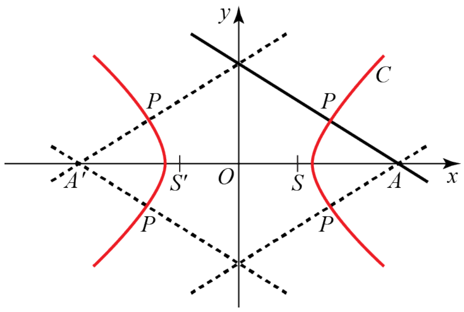
$$y - y_1 = m'(x - x_1)$$

$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

$$by - b^2\tan t = -ax\sin t + a^2\tan t$$

$$ax\sin t + by = (a^2 + b^2)\tan t, \text{ as required.}$$

27 b



A diagram is essential here. Without it, you would be unlikely to see that there are four possible point where $OA = 3OS$
There are two to the right of the y-axis, corresponding to the focus S with coordinates $(ae, 0)$, and two to the left of the y-axis, corresponding to the focus, here marked S', with coordinates $(-ae, 0)$

The x-coordinate of A is given by
$ax\sin t + 0 = (a^2 + b^2)\tan t$

$$x = \frac{a^2+b^2}{a} \times \frac{\tan t}{\sin t} = \frac{a^2+b^2}{a\cos t}$$

Hence $OA = \dfrac{a^2+b^2}{a\cos t}$

Using $b^2 = a^2(e^2-1)$ with $e = \dfrac{3}{2}$

You need to eliminate b from the length OA to obtain a solvable equation in t from the condition $OA = 3AS$

$$b^2 = a^2\left(\frac{9}{4}-1\right) = \frac{5a^2}{4}$$

and $OA = \dfrac{a^2+b^2}{a\cos t} = \dfrac{a^2+\frac{5a^2}{4}}{a\cos t} = \dfrac{9a}{4\cos t}$

As $e = \dfrac{3}{2}$,

$$OS = ae = \frac{3a}{2}$$

$$OA = 3OS$$

$$\frac{9a}{4\cos t} = \frac{9a}{2} \Rightarrow \cos t = \frac{1}{2}$$

$$t = \frac{\pi}{3}, \frac{5\pi}{3}$$

These values give two points P, $(2a, \sqrt{3}b)$ and $(2a, -\sqrt{3}b)$

These are the solutions in the first and fourth quadrants.
From the diagram, by symmetry, there are also solutions in the second and third quadrants giving

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

These correspond to the two points $(-2a, \sqrt{3}b)$ and $(-2a, -\sqrt{3}b)$ where $\cos t = -\dfrac{1}{2}$

The possible values of t are

$$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

28 a $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

$x^2-y^2=a^2 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

This is an hyperbola in which $a = b$

$b^2=a^2(e^2-1)$

For this hyperbola $b^2=a^2$

$a^2=a^2(e^2-1) \Rightarrow 1=e^2-1 \Rightarrow e^2=2$

$e=\sqrt{2}$, as required.

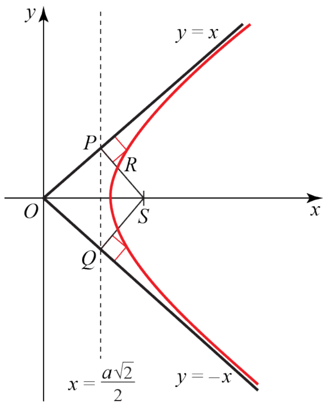
b The coordinates of S are

$(ae,0)=(a\sqrt{2},0)$

An equation of L is

$x=\frac{a}{e}=\frac{a}{\sqrt{2}}=\frac{a\sqrt{2}}{2}$

c



SP is perpendicular to $y=x$, so its gradients is -1

An equation of SP is

$y=-1(x-a\sqrt{2})=-x+a\sqrt{2}$

$y+x=a\sqrt{2}$

SP meets $y=x$ where

$x+x=a\sqrt{2} \Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence P is on the directrix L.

The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$

These formulae are given in the Edexcel formulae booklet. With this hyperbola $a = b$ and the asymptotes are $y = \pm x$

This question is about the intersection of line with the asymptotes. The lines $y = x$ and $y = -x$ are perpendicular to each other and a hyperbola with perpendicular asymptotes is called a rectangular hyperbola. In Module FP 1, you studied another rectangular hyperbola, $xy = c^2$

SQ is perpendicular to $y=-x$, so its gradient is 1

An equation of SQ is

$y=1(x-a\sqrt{2})=x-a\sqrt{2}$

$y=x-a\sqrt{2}$

SQ meets $y=-x$ where $-x=x-a\sqrt{2} \Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence Q is on the directrix L.

Both P and Q lie on the directrix L.

The coordinates of P are $\left(\frac{a\sqrt{2}}{2},\frac{a\sqrt{2}}{2}\right)$

The coordinates of Q are $\left(\frac{a\sqrt{2}}{2},-\frac{a\sqrt{2}}{2}\right)$

28 d $SP: y + x = a\sqrt{2}$ **(1)**

Hyperbola $x^2 - y^2 = a^2$ **(2)**

Form **(1)** $y = a\sqrt{2} - x$ **(3)**

Substitute **(3)** into **(2)**

$$x^2 - (a\sqrt{2} - x)^2 = a^2$$

$$x^2 - 2a^2 + 2\sqrt{2}ax - x^2 = a^2$$

$$2\sqrt{2}ax = 3a^2 \Rightarrow x = \frac{3a}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}a$$

Substituting for x in **(3)**

$$y = a\sqrt{2} - \frac{3\sqrt{2}}{4}a = \frac{\sqrt{2}}{4}a$$

To find the tangent to the hyperbola at R

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

At R

$$\frac{dy}{dx} = \frac{x}{y} = \frac{\frac{3\sqrt{2}}{4}a}{\frac{\sqrt{2}}{4}a} = 3$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{2}}{4}a = 3\left(x - \frac{3\sqrt{2}}{4}a\right) = 3x - \frac{9\sqrt{2}}{4}a$$

$$y = 3x - 2\sqrt{2}a$$

At $x = \frac{a\sqrt{2}}{2}, y = 3\left(\frac{a\sqrt{2}}{2}\right) - 2\sqrt{2}a = -\frac{a\sqrt{2}}{2}$

This is the y-coordinate of Q.

Hence the tangent at R passes through Q.

To find the coordinates of R, you solve equations **(1)** and **(2)** simultaneously.

The coordinates of R are $\left(\frac{3\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a\right)$

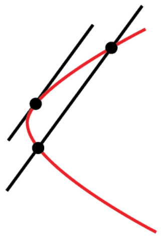
Differentiating the equation of the hyperbola implicitly with respect to x.

This is the equation of the tangent to the hyperbola at R. To establish that R passes through Q, you substitute the x-coordinate of Q into this equation and show that this gives the y-coordinate of Q.

29 Let the equation of the tangent be $y = mx + c$

Eliminating y between $y = mx + c$ and $x^2 - 4y^2 = 4$

$$x^2 - 4(mx+c)^2 = 4$$
$$x^2 - 4m^2x^2 - 8mcx - 4c^2 = 4$$
$$(4m^2 - 1)x^2 + 8mcx + 4(c^2+1) = 0 \quad *$$



If the line was a chord, it would cut the curve in two distinct points and this equation would have a positive discriminant. As the line is a tangent it touches the curve at just one point and this equation has a repeated root. The discriminant is zero.

As the line is a tangent, equation * has repeated roots

'$b^2 - 4ac = 0$'

$$64m^2c^2 - 16(4m^2-1)(c^2+1) = 0$$
$$64m^2c^2 - 64m^2c^2 - 64m^2 + 16c^2 + 16 = 0$$
$$16c^2 = 64m^2 - 16$$
$$c^2 = 4m^2 - 1 \Rightarrow c = \pm\sqrt{(4m^2-1)}$$

The equation of the tangent is

$y = mx \pm \sqrt{(4m^2-1)}$, where $|m| > \frac{1}{2}$, as required.

If $|m| < \frac{1}{2}$, then $\sqrt{(4m^2-1)}$ would be the square root of a negative number and there would be no real answer. The cases $m = \pm\frac{1}{2}$ are interesting. For these values the equations are $y = \pm\frac{1}{2}x$

These are the asymptotes of the hyperbola and do not touch it at any point with finite coordinates. Asymptotes can be thought of as tangents to the curve 'at infinity'.

30 a $x = a\cos t, \quad y = b\sin t$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b\cos t}{a\sin t}$$

For the tangent

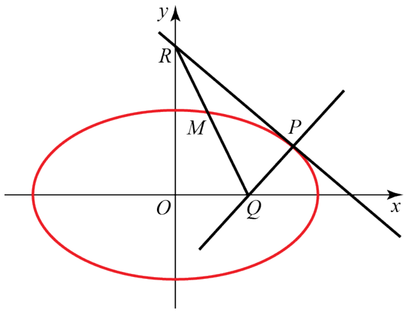
$$y - y_1 = m(x - x_1)$$
$$y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$$
$$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$$
$$ay\sin t + bx\cos t = ab(\sin^2 t + \cos^2 t)$$
$$ay\sin t + bx\cos t = ab$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However the calculation in part **c** will be easier if you simplify the equation at this stage using $\sin^2 t + \cos^2 t = 1$

b As $\frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$, using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin t}{b\cos t}$$
$$y - y_1 = m'(x - x_1)$$
$$y - b\sin t = \frac{a\sin t}{b\cos t}(x - a\cos t)$$
$$by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$$
$$ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$$

30 c



The condition $0 < t < \frac{\pi}{2}$ implies that P is in the first quadrant.

Substituting $y = 0$ into the answer to part **b**

$$ax\sin t = (a^2 - b^2)\sin t\cos t \Rightarrow x = \frac{a^2 - b^2}{a}\cos t$$

You find the x-coordinate of Q by substituting $y = 0$ into the equation you found for the normal in part **b** and solving for x.

The coordinates of Q are $\left(\frac{a^2 - b^2}{a}\cos t, 0\right)$

Substituting $x = 0$ into the answer to part **a**

$$ay\sin t = ab \Rightarrow y = \frac{b}{\sin t}$$

You find the y-coordinate of R by substituting $x = 0$ into the equation you found for the tangent in part **a** and solving for y.

The coordinates of R are $\left(0, \frac{b}{\sin t}\right)$

The coordinates of M are given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{a^2 - b^2}{2a}\cos t, \frac{b}{2\sin t}\right)$$

d If the coordinates of M are (x, y) then $x = \frac{a^2 - b^2}{2a}\cos t \Rightarrow \cos t = \frac{2ax}{a^2 - b^2}$ and

$$y = \frac{b}{2\sin t} \Rightarrow \sin t = \frac{b}{2y}$$

As $\cos^2 t + \sin^2 t = 1$, the locus of

M is $\left(\frac{2ax}{a^2 - b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$, as required.

$x = \frac{a^2 - b^2}{2a}\cos t$ and $y = \frac{b}{2\sin t}$ are the parametric equations of the locus of M. To find the Cartesian equation, you must eliminate t. The form of the answer given in the question gives you a hint that you can use the identity $\cos^2 t + \sin^2 t = 1$ to do this.

31 a To find the equation of the tangent at $(a\sec\theta, b\tan\theta)$

$$x = a\sec\theta, \quad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \frac{dy}{dt} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$y - y_1 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However, the calculation in part **b** will be easier if you simplify the equation at this stage.

$$ay\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = bx - ab\sec\theta$$

$$bx - ay\sin\theta = ab\left(\frac{1-\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta}$$

$$bx - ay\sin\theta = ab\cos\theta \quad \textbf{(1)}$$

To find the equation of the normal at $(a\sec\theta, b\tan\theta)$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{a\sin\theta}{b}$$

$$y - y_1 = m'(x - x_1)$$

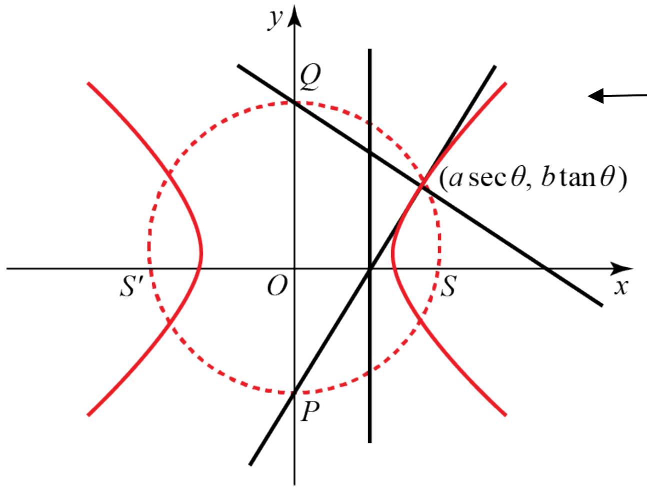
$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

When you multiply the brackets out, $\sin\theta\sec\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\tan\theta$$

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta \quad \textbf{(2)}$$

31 b



This problem will be solved using the property that the angle in a semi-circle is a right angle and you need to show that PS and QS are perpendicular. All five of the points, P, Q, $(a\sec\theta, b\tan\theta)$ and the two foci lie on the same circle.

Substitute $x = 0$ into **(1)**
$-ay\sin\theta = ab\cos\theta \Rightarrow y = -b\cot\theta$

To find the coordinates of P, you substitute $x = 0$ into the equation of the tangent found in part **a**.

The coordinates of the P are $(0, -b\cot\theta)$
Substitute $x = 0$ into **(2)**

$$by = (a^2 + b^2)\tan\theta \Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$$

To find the coordinates of Q, you substitute $x = 0$ into the equation of the normal found in part **a**.

The coordinates of Q are $\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

The focus S has coordinates $(ae, 0)$

The gradient of PS is given by $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b\cot\theta - 0}{0 - ae} = \frac{b}{ae}\cot\theta$

The gradient of QS is given by

$$m' = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{a^2+b^2}{b}\tan\theta - 0}{0 - ae} = \frac{-(a^2+b^2)}{abe}\tan\theta$$

$$mm' = \frac{b}{ae}\cot\theta \times -\frac{a^2+b^2}{abe}\tan\theta = -\frac{a^2+b^2}{a^2e^2}$$

The formula for the eccentricity is
$b^2 = a^2(e^2 - 1)$
$b^2 = a^2e^2 - a^2 \Rightarrow a^2e^2 = a^2 + b^2$

Hence $mm' = -\frac{a^2+b^2}{a^2e^2} = -\frac{a^2+b^2}{a^2+b^2} = -1$

So PS is perpendicular to QS and $\angle PSQ = 90°$
By the converse of the theorem that the angle in a semi-circle is a right angle, the circle described on PQ as diameter passes through the focus S.
By symmetry, the circle also passes through the focus S'.

There is no need to repeat the calculations for PS' and QS'. It is evident from the diagram that the whole diagram is symmetrical about the y-axis, so, if the circle passes through S, it passes through S'. It is quite acceptable to appeal to symmetry to complete your proof.

32 a $\cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2}$

When $x = \ln k$

$$\frac{e^{2x} + e^{-2x}}{2} = \frac{e^{2\ln k} + e^{-2\ln k}}{2}$$

$$= \frac{e^{\ln k^2} + e^{\ln\left(\frac{1}{k^2}\right)}}{2}$$

$$= \frac{k^2 + \frac{1}{k^2}}{2}$$

$$= \frac{k^4 + 1}{2k^2} \text{ as required}$$

b $f(x) = px - \tanh 2x$

$$f'(x) = p - 2\operatorname{sech}^2 2x$$

$$= p - \frac{2}{\cosh^2 2x}$$

At $x = \ln 2$, $f'(x) = 0$, so:

$$p - \frac{2}{\cosh^2(2\ln 2)} = 0$$

From part **a** with $k = 2$:

$$\cosh(2\ln 2) = \frac{2^4 + 1}{2 \times 2^2}$$

$$= \frac{17}{8}$$

Therefore:

$$p - \frac{2}{\left(\frac{17}{8}\right)^2} = 0$$

$$p = \frac{128}{289}$$

33 a $y = -x + \tanh 4x$

$$\frac{dy}{dx} = -1 + 4\operatorname{sech}^2 4x = 0$$

$$\operatorname{sech}^2 4x = \frac{1}{4} \Rightarrow \cosh^2 4x = 4$$

$$\cosh 4x = 2$$

$$4x = \operatorname{arcosh} 2 = \ln\left(2+\sqrt{3}\right)$$

$$x = \frac{1}{4}\ln\left(2+\sqrt{3}\right)$$

As $\cosh x \geqslant 1$ for all real x, $\cosh 4x = -2$ is impossible.

For $x \geqslant 0$, there is only one value of x which gives a stationary value. The question tells you that the curve has a maximum point so, in this question, you need not show that this point is a maximum by, for example, examining the second derivative.

b $\tanh^2 4x = 1 - \operatorname{sech}^2 4x = 1 - \frac{1}{4} = \frac{3}{4}$

You need a value for $\tanh 4x$ and this is easiest found using the hyperbolic identity $\operatorname{sech}^2 x = 1 - \tanh^2 x$.

As $x \geqslant 0$, $\tanh 4x = \frac{\sqrt{3}}{2}$

At $x = \frac{1}{4}\ln\left(2+\sqrt{3}\right)$

$$y = -x + \tanh 4x = -\frac{1}{4}\ln\left(2+\sqrt{3}\right) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left(2\sqrt{3} - \ln\left(2+\sqrt{3}\right)\right), \text{ as required.}$$

34

$$x = \frac{a}{\sinh\theta} = a(\sinh\theta)^{-1}$$

$$\frac{dx}{d\theta} = -a(\sinh\theta)^{-2}\cosh\theta = -\frac{a\cosh\theta}{\sinh^2\theta}$$

When substituting remember to substitute for the dx as well as the rest of the integral.

$$\int \frac{1}{x\sqrt{(x^2+a^2)}}\,dx = \int \frac{1}{\frac{a}{\sinh\theta}\sqrt{\left(\frac{a^2}{\sinh^2\theta}+a^2\right)}} \times \frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{\frac{-a\cosh\theta}{\sinh^2\theta}}{\frac{a^2\sqrt{1+\sinh^2\theta}}{\sinh^2\theta}}\,d\theta = \frac{-1}{a}\int \frac{\cosh\theta}{\cos\theta}\,d\theta$$

Use $1 + \sinh^2\theta = \cosh^2\theta$ to simplify this expression.

$$= -\frac{1}{a}\int \frac{\cosh\theta}{\cosh\theta}\,d\theta = -\frac{1}{a}\int 1\,d\theta$$

$$= -\frac{1}{a}\theta + \text{constant}$$

$$= -\frac{1}{a}\operatorname{arsinh}\left(\frac{a}{x}\right) + \text{constant, as required.}$$

As $x = \frac{a}{\sinh\theta}$, then $\sinh\theta = \frac{a}{x}$ and $\theta = \operatorname{arsinh}\left(\frac{a}{x}\right)$.

35 a Let $y = \text{artanh}\, x$

$\tanh y = x$

Differentiate implicitly with respect to x

$$\text{sech}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

$$= \frac{1}{1-x^2}, \text{ as required}$$

To differentiate a function f(y) with respect to x you use a version of the chain rule $\frac{d}{dx}(f(y)) = f'(y) \times \frac{dy}{dx}$.

b Using integration by parts and the result in part **a**

$$\int \text{artanh}\, x\, dx = \int 1 \times \text{artanh}\, x\, dx$$

$$= x\, \text{artanh}\, x - \int \frac{x}{1-x^2} dx$$

$$= x\, \text{artanh}\, x + \tfrac{1}{2} \ln(1-x^2) + A$$

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = \text{artanh}\, x$ and $\frac{dv}{dx} = 1$. You know $\frac{du}{dx}$ from part **a**.

This solution uses the result $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. So $\int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$ and you multiply this by $-\frac{1}{2}$ to complete the solution. This is a question where there are a number of possible alternative forms of the answer.

36 a Let $y = \text{arsinh}\, x$ then $x = \sinh y = \frac{e^y - e^{-y}}{2}$

$$2x = e^y - e^{-y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x + \sqrt{(4x^2 - 4)}}{2}$$

$$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides, $y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $y = \text{arsinh}\, x$

$\sinh y = x$

Differentiating implicitly with respect to x

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2 \Rightarrow \cosh y = \sqrt{(1+x^2)}$$

Hence $\frac{d}{dx}(\text{arsinh}\, x) = \frac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}}$, as required.

arsinh x is an increasing function of x for all x. So its gradient is always positive and you need not consider the negative square root.

36 c $y = (\text{arsinh}\, x)^2$

$$\frac{dy}{dx} = 2\text{arsinh}\, x\,(1+x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2\,(1+x^2)^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + 2\text{arsinh}\, x \times \left(-\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{3}{2}}$$

$$= 2(1+x^2)^{-1} - 2x\text{arsinh}\, x\,(1+x^2)^{-\frac{3}{2}}$$

You use the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 1\ 2\text{arsinh}\, x$ and $v = (1+x^2)^{-\frac{1}{2}}$.

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2$$

$$= (1+x^2)\left(2(1+x^2)^{-1} - 2\text{arsinh}\, x(1+x^2)^{-\frac{3}{2}}\right) + x \times 2\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} - 2$$

$$= 2 - 2x\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} + 2x\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} - 2$$

$= 0$, as required.

d

$$\int_0^1 \text{arsinh}\, x\, dx = \int_0^1 1 \times \text{arsinh}\, x\, dx$$

$$= \left[x\text{arsinh}\, x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{(1+x^2)}}\, dx$$

$$= \text{arsinh}\, 1 - \left[\sqrt{(1+x^2)}\right]_0^1$$

$$= \ln\left(1+\sqrt{2}\right) - \sqrt{2} + 1$$

$\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) = \frac{1}{2} \times 2x \times (1+x^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{(1+x^2)}}$

so $\int \frac{x}{\sqrt{(1+x^2)}}\, dx = \sqrt{(1+x^2)}$.

37 a

$$4x^2 + 4x + 5 = (px+q)^2 + r$$

$$= p^2x^2 + 2pqx + q^2 + r$$

Equating coefficients of x^2

$4 = p^2 \Rightarrow p = 2$

The conditions of the question allow $p = -2$ as an answer, but the negative sign would make the integrals following awkward, so choose the positive root.

Equating coefficients of x

$4 = 2pq = 4q \Rightarrow q = 1$

Equating constant coefficients

$5 = q^2 + r = 1 + r \Rightarrow r = 4$

$p = 2, q = 1, r = 4$

37 b $\int \frac{1}{4x^2+4x+5}\,dx = \int \frac{1}{(2x+1)^2+4}\,dx$

Let $2x+1 = 2\tan\theta$

$$2\frac{d}{d\theta} = 2\sec^2\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$$

$$\int \frac{1}{(2x+1)^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{1}{4\cancel{\sec^2\theta}}(\cancel{\sec^2\theta})\,d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{2}\right) + C$$

If you know a formula of the type $\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right)$, or you are confident at writing down integrals by inspection, you may be able to find this integral without working. It is, however, very easy to make errors with the constant and get, for example, the common error $\frac{1}{2}\arctan\left(\frac{2x+1}{2}\right)+C$.

c $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \int \frac{2}{\sqrt{\left((2x+1)^2+4\right)}}\,dx$

As in part **b**, you may be able to write down this integral without working.

Let $2x+1 = 2\sinh\theta$

$$2\frac{dx}{d\theta} = 2\cosh\theta \Rightarrow \frac{dx}{d\theta} = \cosh\theta$$

$$\int \frac{2}{\sqrt{\left((2x+1)^2+4\right)}}\,dx = \int \frac{2}{\sqrt{(4\sinh^2\theta+4)}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{2}{2\cosh\theta}(\cosh\theta)\,d\theta = \int 1\,d\theta$$

$$= \theta + C = \operatorname{arsinh}\left(\frac{2x+1}{2}\right) + C$$

Using $\operatorname{arsinh} x = \ln\left(x+\sqrt{(x^2+1)}\right)$

$$\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{(2x+1)^2}{4}+1\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{4x^2+4x+1+4}{4}\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\frac{1}{2}\sqrt{(4x^2+4x+5)}\right]+C$$

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]-\ln 2+C$$

$-\ln 2+$ an arbitrary constant is another arbitrary constant.

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]+k, \text{ as required.}$$

38 Solve the integral as follows:

$$\int \frac{x+2}{\sqrt{4x^2+9}}\,dx$$

$$= \int \frac{x}{\sqrt{4x^2+9}}\,dx + \int \frac{2}{\sqrt{4x^2+9}}\,dx$$

Then the first integral is a standard integral of the form $kf'(x)(f(x))^n$, and we can easily see that its value is $\frac{\sqrt{4x^2+9}}{4}$, as the derivative of $\sqrt{4x^2+9}$ is $\frac{4x}{\sqrt{4x^2+9}}$. For the second integral we make a substitution: if $t = \frac{2}{3}x$, then:

$$\int \frac{2}{\sqrt{4x^2+9}}\,dx$$

$$= \int \frac{2}{\sqrt{4\times\frac{9}{4}t^2+9}}\,\frac{3}{2}dt$$

$$= \int \frac{1}{\sqrt{t^2+1}}\,dt = \text{arsinh}\ t = \text{arsinh}\ \frac{2}{3}x$$

Then we can conclude that the integral of the function is $\frac{4x^2+9}{4} + \text{arsinh}\ \frac{2}{3}x + C$.

39 Since $(x-2)^2 = x^2-4x+4$, we can rewrite the integral as $= \int_2^5 \frac{1}{2\sqrt{\frac{(x-2)^2}{4}+1}}\,dx$. So if we put $t = \frac{x-2}{2}$ we get that the integral is:

$$\int_0^{\frac{3}{2}} \frac{1}{2\sqrt{t^2+1}}\,2\ dt$$

$$= \int_0^{\frac{3}{2}} \frac{1}{\sqrt{t^2+1}}\,dt$$

$$= \text{arsinh}\ \tfrac{3}{2}$$

40 $\int x \operatorname{arcosh} x \, dx = \frac{x^2}{2} \operatorname{arcosh} x - \int \frac{x^2}{2\sqrt{(x^2-1)}} dx$

This solution uses integration by parts, $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = \operatorname{arcosh} x$ and $\frac{dv}{dx} = x$. There are other possible approaches to this question, for example, substituting $u = \operatorname{arcosh} x$.

To find the remaining integral, let $x = \cosh \theta$.

$$\frac{dx}{d\theta} = \sinh \theta$$

$$\int \frac{x^2}{2\sqrt{(x^2-1)}} dx = \int \frac{\cosh^2 \theta}{2\sqrt{\cosh^2 \theta - 1}} \left(\frac{dx}{d\theta}\right) d\theta$$

$$= \int \frac{\cosh^2 \theta}{2\sinh \theta} \sinh \theta \, d\theta = \frac{1}{2} \int \cosh^2 \theta \, d\theta$$

$$= \frac{1}{4} \int (\cosh 2\theta + 1) \, d\theta$$

Using the identity $\cosh 2\theta = 2\cosh^2 \theta - 1$.

$$= \frac{\sinh 2\theta}{8} + \frac{\theta}{4} = \frac{\sinh \theta \cosh \theta}{4} + \frac{\theta}{4}$$

$$= \frac{\left[\sqrt{(x^2-1)}\right] x}{4} + \frac{1}{4} \operatorname{arcosh} x$$

$\sinh \theta = \sqrt{(\cosh^2 \theta - 1)} = \sqrt{(x^2-1)}$

Hence the area, A, of R is given by

$$A = \left[\frac{x^2}{2} \operatorname{arcosh} x - \frac{1}{4} x\sqrt{(x^2-1)} - \frac{1}{4} \operatorname{arcosh} x\right]_1^2$$

$$= \left[\left(\frac{x^2}{2} - \frac{1}{4}\right) \operatorname{arcosh} x - \frac{1}{4} x\sqrt{(x^2-1)}\right]_1^2$$

$$= \left[\frac{7}{4} \operatorname{arcosh} 2 - \frac{\sqrt{3}}{2}\right] - [0]$$

As $\operatorname{arcosh} 1 = 0$ and $\sqrt{(1^2-1)} = 0$, both terms are zero at the lower limit.

$$= \frac{7}{4} \ln\left(2 + \sqrt{3}\right) - \frac{\sqrt{3}}{2}, \text{ as required.}$$

41 a $I_n = \int \sec^n x \, dx, \quad n \geq 0$

Use integration by parts with:

$$u = \sec^{n-2} x \Rightarrow \frac{du}{dx} = (n-2)\sec^{n-2} x \tan x$$

and

$$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$\begin{aligned} I_n &= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \, dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x - (n-2)\int (\sec^n x - \sec^{n-2} x) dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x \, dx + (n-2)\int \sec^{n-2} x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2} \end{aligned}$$

$$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$ as required

b $I_4 = \int \sec^4 x \, dx$

Using the reduction formula from part a:

$$\begin{aligned} 3I_4 &= \sec^2 x \tan x + 2\int \sec^2 x \, dx \\ &= \sec^2 x \tan x + 2\tan x + c \end{aligned}$$

$$I_4 = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + c$$

42 a $I_n = \int_0^{\frac{\pi}{4}} x^n \cos x \, dx, \; n \geq 0$

Use integration by parts with:

$u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

and

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$$I_n = \left[x^n \sin x\right]_0^{\frac{\pi}{4}} - n\int_0^{\frac{\pi}{4}} x^{n-1} \sin x \, dx$$

Use integration by parts again with:

$u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$

and

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$$I_n = \left[x^n \sin x\right]_0^{\frac{\pi}{4}} - n\left(\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{4}} - (n-1)\int_0^{\frac{\pi}{4}} x^{n-2}(-\cos x)\, dx\right)$$

$$= \left[x^n \sin x\right]_0^{\frac{\pi}{4}} + n\left[x^{n-1}\cos x\right]_0^{\frac{\pi}{4}} - n(n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\cos x \, dx$$

$$= \left[x^n \sin x\right]_0^{\frac{\pi}{4}} + n\left[x^{n-1}\cos x\right]_0^{\frac{\pi}{4}} - n(n-1)I_{n-2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)^n + \frac{n}{\sqrt{2}}\left(\frac{\pi}{4}\right)^{n-1} - n(n-1)I_{n-2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)^{n-1}\left(\frac{\pi}{4} + n\right) - n(n-1)I_{n-2} \text{ as required}$$

42 b $I_4 = \int_0^{\frac{\pi}{4}} x^4 \cos x \, dx, n \geq 0$

Using the reduction formula from part a:

$$I_4 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)^{4-1}\left(\frac{\pi}{4}+4\right) - 4(4-1)I_{4-2}$$

$$= \frac{\pi^3}{64\sqrt{2}}\left(\frac{\pi}{4}+4\right) - 12I_2$$

And:

$$I_2 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)^{2-1}\left(\frac{\pi}{4}+2\right) - 2(2-1)I_{2-2}$$

$$= \frac{1}{\sqrt{2}} \times \frac{\pi}{4}\left(\frac{\pi}{4}+2\right) - 2I_0$$

Calculating I_0:

$$I_0 = \int_0^{\frac{\pi}{4}} x^0 \cos x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}$$

Putting everything together:

$$I_4 = \frac{\pi^3}{64\sqrt{2}}\left(\frac{\pi}{4}+4\right) - 12\left(\frac{1}{\sqrt{2}} \times \frac{\pi}{4}\left(\frac{\pi}{4}+2\right) - 2 \times \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi^3}{64\sqrt{2}}\left(\frac{\pi}{4}+4\right) - \frac{3\pi}{\sqrt{2}}\left(\frac{\pi}{4}+2\right) + 12\sqrt{2}$$

$$= 0.0471197...$$

$$= 0.0471 \text{ (4 d.p.)}$$

43 a $I_n = \int_0^a x^n (a-x)^{\frac{1}{3}} \, dx, \; n \geq 0, \; a > 0$

Use integration by parts with:

$u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

and

$\frac{dv}{dx} = (a-x)^{\frac{1}{3}} \Rightarrow v = -\frac{3}{4}(a-x)^{\frac{4}{3}}$

$$I_n = \left[-\frac{3}{4}x^n (a-x)^{\frac{4}{3}}\right]_0^a - \int_0^a \left(-\frac{3}{4}nx^{n-1}(a-x)^{\frac{4}{3}}\right) dx$$

$$= -\frac{3}{4}\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + \frac{3}{4}n\int_0^a x^{n-1}(a-x)^{\frac{4}{3}} \, dx$$

$$= -\frac{3}{4}\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + \frac{3}{4}n\int_0^a x^{n-1}(a-x)(a-x)^{\frac{1}{3}} \, dx$$

$$= -\frac{3}{4}\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + \frac{3}{4}n\left(\int_0^a ax^{n-1}(a-x)^{\frac{1}{3}} \, dx - \int_0^a x^n (a-x)^{\frac{1}{3}} \, dx\right)$$

$$= -\frac{3}{4}\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + \frac{3}{4}an\int_0^a x^{n-1}(a-x)^{\frac{1}{3}} \, dx - \frac{3}{4}n\int_0^a x^n (a-x)^{\frac{1}{3}} \, dx$$

$$= -\frac{3}{4}\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + \frac{3}{4}anI_{n-1} - \frac{3}{4}nI_n$$

$$\frac{4}{3}I_n = -\left[x^n (a-x)^{\frac{4}{3}}\right]_0^a + anI_{n-1} - nI_n$$

$$\frac{4}{3}I_n + nI_n = -\left[\left(a^n (a-a)^{\frac{4}{3}}\right) - \left(0^n (a-0)^{\frac{4}{3}}\right)\right] + anI_{n-1}$$

$$I_n\left(\frac{4}{3} + n\right) = anI_{n-1}$$

$$I_n\left(\frac{4+3n}{3}\right) = anI_{n-1}$$

$$I_n = \frac{3an}{4+3n}I_{n-1} \text{ as required}$$

43 b $I_2 = \frac{27}{49}a^{\frac{4}{3}}$

Using the reduction formula from part **a**:

$$I_2 = \frac{3a(2)}{4+3(2)}I_1$$
$$= \frac{3}{5}aI_1$$
$$I_1 = \frac{3a(1)}{4+3(1)}I_0$$
$$= \frac{3}{5}aI_0$$

Integrating directly:

$$I_0 = \int_0^a (a-x)^{\frac{1}{3}}\,\mathrm{d}x$$
$$= -\frac{3}{4}\left[(a-x)^{\frac{4}{3}}\right]_0^a$$
$$= -\frac{3}{4}\left[(a-a)^{\frac{4}{3}} - (a-0)^{\frac{4}{3}}\right]$$
$$= \frac{3}{4}a^{\frac{4}{3}}$$

Putting everything together:

$$I_2 = \frac{3}{5}a \times \frac{3}{7}a \times \frac{3}{4}a^{\frac{4}{3}}$$
$$= \frac{27}{140}a^{\frac{10}{3}}$$

Since $I_2 = \frac{27}{49}a^{\frac{4}{3}}$

$$\frac{27}{140}a^{\frac{10}{3}} = \frac{27}{49}a^{\frac{4}{3}}$$
$$a^2 = \frac{140}{49}$$
$$a = \frac{\sqrt{140}}{7}$$

44 a $y = \left(ax^3\right)^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{3}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}$

Using $s = \int_{x_A}^{x_B}\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x$ gives:

$$s = \int_0^4\left(1+\left(\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x$$

$$= \int_0^4\left(1+\frac{9}{4}ax\right)^{\frac{1}{2}}\mathrm{d}x$$

Let $u = 1+\frac{9}{4}ax \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{9}{4}a \Rightarrow \mathrm{d}x = \frac{4}{9a}$

when $x = 0$, $u = 1$

when $x = 4$, $u = 1 + 9a$

$$\int_0^4\left(1+\frac{9}{4}ax\right)^{\frac{1}{2}}\mathrm{d}x = \frac{4}{9a}\int_1^{1+9a}u^{\frac{1}{2}}\mathrm{d}u$$

$$= \frac{4}{9a}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{1+9a}$$

$$= \frac{8}{27a}\left[(1+9a)^{\frac{3}{2}}-1\right]$$

b $s = 16$, therefore:

$$\frac{8}{27a}\left[(1+9a)^{\frac{3}{2}}-1\right] = 16$$

$$(1+9a)^{\frac{3}{2}}-1 = 54a$$

$$(1+9a)^{\frac{3}{2}} = 54a+1$$

$$(1+9a)^3 = (54a+1)^2$$

$$\left(1+18a+81a^2\right)(1+9a) = 2916a^2+108a+1$$

$$1+27a+243a^2+729a^3 = 2916a^2+108a+1$$

$$729a^3-2673a^2-81a = 0$$

$$81a\left(9a^2-33a-1\right) = 0$$

Since $a \neq 0$,

$$a = \frac{33\pm\sqrt{33^2-4(9)(-1)}}{2(9)}$$

$$\left(=\frac{11\pm5\sqrt{5}}{6}\right)$$

$a = 3.696...$ or $a = -0.0300...$

$a > 0$, therefore:

$a = 3.6967$ (4 d.p.)

45 a $y = 2x + 16$

$x = \frac{1}{2}y^2 - 8$

$\frac{dx}{dy} = y$

Using $s = \int_{y_A}^{y_B} \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}} dy$ gives:

$s = \int_0^3 \left(1 + y^2\right)^{\frac{1}{2}} dy$ as required

b $s = \int_0^3 \left(1 + y^2\right)^{\frac{1}{2}} dy$

Let $y = \sinh u \Rightarrow dy = \cosh u du$

When $y = 0$, $u = 0$

When $y = 3$, $u = \text{arsinh } 3$

$$s = \int_0^3 \left(1 + y^2\right)^{\frac{1}{2}} dy$$

$$= \int_0^{\text{arsinh}3} \left(1 + \sinh^2 u\right)^{\frac{1}{2}} \cosh u du$$

$$= \int_0^{\text{arsinh}3} \cosh^2 u du$$

$$= \frac{1}{2}\int_0^{\text{arsinh}3} \left(1 + \cosh 2u\right) du$$

$$= \frac{1}{2}\left[u + \frac{1}{2}\sinh 2u\right]_0^{\text{arsinh}3}$$

$$= \frac{1}{2}\left[u + \sinh u \cosh u\right]_0^{\text{arsinh}3}$$

$$= \frac{1}{2}\left[u + \sinh u\sqrt{1 + \sinh^2 u}\right]_0^{\text{arsinh}3}$$

$$= \frac{1}{2}\text{arsinh}3 + \frac{1}{2}\left(3\sqrt{10}\right)$$

since $\text{arsinh } x = \ln\left(x + \sqrt{1 + x^2}\right)$

$$s = \frac{1}{2}\ln\left(3 + \sqrt{10}\right) + \frac{3}{2}\sqrt{10}$$

46 $x = t^2 - 1 \Rightarrow \dfrac{dx}{dt} = 2t$

$y = \dfrac{1}{3}t^3 - 2 \Rightarrow \dfrac{dy}{dt} = t^2$

Using $s = \displaystyle\int_{t_A}^{t_B} \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right)^{\frac{1}{2}} dt$ gives:

$$s = \int_0^2 \left((2t)^2 + (t^2)^2 \right)^{\frac{1}{2}} dt$$

$$= \int_0^2 \left(4t^2 + t^4 \right)^{\frac{1}{2}} dt$$

$$= \int_0^2 t\left(4 + t^2 \right)^{\frac{1}{2}} dt$$

let $u = 4 + t^2 \Rightarrow du = 2t dt$
when $t = 0$, $u = 4$
when $t = 2$, $u = 8$

$$s = \frac{1}{2}\int_0^2 u^{\frac{1}{2}} du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} \right]_4^8$$

$$= \frac{1}{3}\left(8^{\frac{3}{2}} - 4^{\frac{3}{2}} \right)$$

$$= \frac{8^{\frac{3}{2}} - 8}{3}$$

47 a For polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

The curve $r = \theta$ can then be expressed using the parametric equations:

$x = r\cos\theta$

$y = \theta\sin\theta$

Differentiating:

$$\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$$

$$\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$$

Therefore:

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2 \\ &= (\cos^2\theta - 2\theta\cos\theta\sin\theta + \theta^2\sin^2\theta) + (\sin^2\theta + 2\theta\cos\theta\sin\theta + \theta^2\cos^2\theta) \\ &= (\cos^2\theta + \sin^2\theta) + (2\theta\cos\theta\sin\theta - 2\theta\cos\theta\sin\theta) + \theta^2(\sin^2\theta + \cos^2\theta) \\ &= 1 + \theta^2\end{aligned}$$

Using $s = \int_{\theta_A}^{\theta_B} \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{\frac{1}{2}} d\theta$ gives:

$$W = \int_0^{4\pi} \sqrt{1+\theta^2}\, d\theta$$

Let $\theta = \tan x \Rightarrow d\theta = \sec^2 x\, dx$

when $\theta = 0$, $x = 0$

when $\theta = 4\pi$, $x = \arctan 4\pi$

$$W = \int_0^{4\pi} \left(1 + \tan^2 x\right)^{\frac{1}{2}} \sec^2 x\, dx$$

47 b $I_n = \int \sec^n x \, dx$

Use integration by parts with:

$u = \sec^{n-2} x \Rightarrow \frac{du}{dx} = (n-2)\sec^{n-2} x \tan x$

and

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \, dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1) \, dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int (\sec^n x - \sec^{n-2} x) dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x \, dx + (n-2)\int \sec^{n-2} x \, dx$$
$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$
$$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$
$$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$
$$I_n = \frac{\sec^{n-2} x \tan x + (n-2) I_{n-2}}{n-1}$$

Since $W = \int_0^{\arctan 4\pi} \sec^3 x \, dx$ using the reduction formula with $n = 3$ gives:

$$W = \frac{[\sec x \tan x]_0^{\arctan 4\pi} + \int_0^{\arctan 4\pi} \sec x \, dx}{2}$$
$$= \frac{[\sec x \tan x]_0^{\arctan 4\pi} + [\ln|\tan x + \sec x|]_0^{\arctan 4\pi}}{2}$$
$$= 80.82\ldots$$
$$= 80.8 \text{ (3 s.f.)}$$

48 $x = 2t^{\frac{1}{2}} \Rightarrow \dfrac{dx}{dt} = t^{-\frac{1}{2}}$

$y = 1 - t \Rightarrow \dfrac{dy}{dt} = -1$

Using $S = 2\pi\displaystyle\int_{t_A}^{t_B} x\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{\frac{1}{2}} dt$ gives:

$$S = 2\pi\int_1^4 2t^{\frac{1}{2}}\left(\left(t^{-\frac{1}{2}}\right)^2 + (-1)^2\right)^{\frac{1}{2}} dt$$

$$= 4\pi\int_1^4 t^{\frac{1}{2}}\left(\frac{1}{t} + 1\right)^{\frac{1}{2}} dt$$

$$= 4\pi\int_1^4 (1+t)^{\frac{1}{2}} dt$$

Let $u = 1 + t \Rightarrow du = dt$
When $t = 1$, $u = 2$
When $t = 4$, $u = 5$

$$S = 4\pi\int_2^5 u^{\frac{1}{2}} du$$

$$= 4\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^5$$

$$= \frac{8}{3}\pi\left[5^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= \frac{8\pi\left(5\sqrt{5} - 2\sqrt{2}\right)}{3}$$

49 a $y=\left(a-x^2\right)^{\frac{1}{2}},\ -1\le x\le 1$

$$\frac{dy}{dx}=-x\left(a-x^2\right)^{-\frac{1}{2}}$$
$$=-\frac{x}{\left(a-x^2\right)^{\frac{1}{2}}}$$

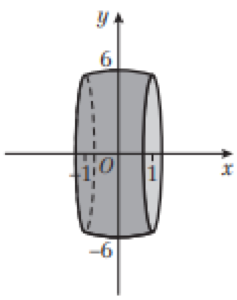
Using $S=2\pi\int_{x_A}^{x_B} y\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}dx$ gives:

$$S=2\pi\int_{-1}^{1}\left(a-x^2\right)^{\frac{1}{2}}\left(1+\frac{x^2}{a-x^2}\right)^{\frac{1}{2}}dx$$
$$=2\pi\int_{-1}^{1}\left(a-x^2\right)^{\frac{1}{2}}\left(\frac{a}{a-x^2}\right)^{\frac{1}{2}}dx$$
$$=2\pi a^{\frac{1}{2}}\int_{-1}^{1}\left(a-x^2\right)^{\frac{1}{2}}\left(\frac{1}{a-x^2}\right)^{\frac{1}{2}}dx$$
$$=2\pi a^{\frac{1}{2}}\int_{-1}^{1}dx$$
$$=2\pi a^{\frac{1}{2}}\left[x\right]_{-1}^{1}$$

the surface has an area of 24π and since the curve has equal area on either side of the axes:

$$2\pi a^{\frac{1}{2}}\left[x\right]_0^1=12\pi$$
$$2\pi a^{\frac{1}{2}}=12\pi$$
$$a^{\frac{1}{2}}=6$$
$$a=36$$

b



50 For polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

The curve $r = \sqrt{\cos 2\theta}$ can then be expressed using the parametric equations:

$x = \sqrt{\cos 2\theta}\cos\theta$

$y = \sqrt{\cos 2\theta}\sin\theta$

Square both sides to ease differentiation:

$x^2 = \cos 2\theta \cos^2\theta$

$y^2 = \cos 2\theta \sin^2\theta$

Differentiate with respect to parameter θ:

$$2x\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right) = \cos 2\theta \times 2\cos\theta(-\sin\theta) + (-2\sin 2\theta)\cos^2\theta$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{1}{x}\cos\theta(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)$$

$$= -\frac{1}{x}\cos\theta\sin 3\theta$$

$$2y\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right) = \cos 2\theta \times 2\sin\theta(\cos\theta) + (-2\sin 2\theta)\sin^2\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{1}{y}\sin\theta(\cos 2\theta\cos\theta - \sin 2\theta\sin\theta)$$

$$= -\frac{1}{y}\sin\theta\cos 3\theta$$

Then:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{1}{x^2}\cos^2\theta\sin^2 3\theta + \frac{1}{y^2}\sin^2\theta\cos^2 3\theta$$

$$= \frac{\cos^2\theta\sin^2 3\theta}{\cos 2\theta\cos^2\theta} + \frac{\sin^2\theta\cos^2 3\theta}{\cos 2\theta\sin^2\theta}$$

$$= \frac{\sin^2 3\theta}{\cos 2\theta} + \frac{\cos^2 3\theta}{\cos 2\theta}$$

$$= \frac{1}{\cos 2\theta}$$

Using $S = 2\pi\int_{x_A}^{x_B} y\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x$ gives:

$$S = 2\pi\int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta}\sin\theta \times \sqrt{\frac{1}{\cos 2\theta}}\,\mathrm{d}x$$

$$= 2\pi\int_0^{\frac{\pi}{4}} \sin\theta\,\mathrm{d}x$$

$$= 2\pi\left[-\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\pi\left(2\pi - \frac{1}{\sqrt{2}}\right)$$

51 $y = e^x,\ 0 \le x \le 1$

$$\frac{dy}{dx} = e^x$$

Using $S = 2\pi\int_{x_A}^{x_B} y\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$ gives:

$$S = 2\pi\int_0^1 e^x\left(1+e^{2x}\right)^{\frac{1}{2}} dx$$

Let $u = e^x \Rightarrow du = e^x dx$
When $x = 0$, $u = 1$
When $x = 1$, $u = e$

$$S = 2\pi\int_1^e \left(1+u^2\right)^{\frac{1}{2}} du$$

Let $u = \sinh v \Rightarrow du = \cosh v\ dv$
When $u = 1$, $v = \text{arsinh } 1$
When $u = e$, $v = \text{arsinh } e$

$$S = 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(e)} (1+\sinh^2 v)^{\frac{1}{2}} \cosh v\ dv$$

$$= 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(e)} \cosh^2 v\ dv$$

$$= 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(e)} \left(\frac{1}{2}+\frac{1}{2}\cosh 2v\right) dv$$

$$= 2\pi\left[\frac{1}{2}v+\frac{1}{4}\sinh 2v\right]_{\text{arsinh}(1)}^{\text{arsinh}(e)}$$

Express $\sinh 2v$ in terns of $\sinh v$ to allow the substitution of limits:

$$S = 2\pi\left[\frac{1}{2}v+\frac{1}{4}\times 2\sinh v\cosh v\right]_{\text{arsinh}(1)}^{\text{arsinh}(e)}$$

$$= \pi\left[v+\sinh v\cosh v\right]_{\text{arsinh}(1)}^{\text{arsinh}(e)}$$

$$= \pi\left[v+\sinh v\sqrt{1+\sinh^2 v}\right]_{\text{arsinh}(1)}^{\text{arsinh}(e)}$$

$$= \pi\left[\left(\text{arsinh}(e)+e\sqrt{1+e^2}\right)-\left(\text{arsinh}(1)+1\sqrt{1+1^2}\right)\right]$$

Since $\text{arsinh } x = \ln\left(x+\sqrt{1+x^2}\right)$,

$$S = \pi\left[\ln\left(e+\sqrt{1+e^2}\right)+e\sqrt{1+e^2}-\ln\left(1+\sqrt{2}\right)-\sqrt{2}\right]$$

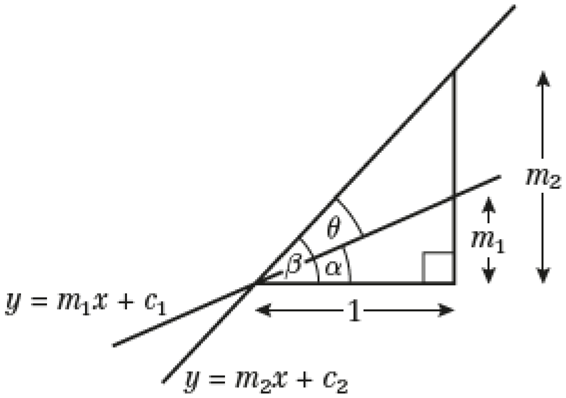
$$= \pi\left[\ln\left(\frac{e+\sqrt{1+e^2}}{1+\sqrt{2}}\right)+e\sqrt{1+e^2}-\sqrt{2}\right]$$

$$= 22.9427\ldots$$

$$= 22.943 \text{ (3 d.p.)}$$

Challenge

1 a



Using the identity for tan($A \pm B$):

$\tan\theta = \tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \dfrac{m_2 - m_1}{1+m_1m_2}$ as required.

1 b Assuming the point P has coordinates ($a\cos t$, $b\sin t$), the gradient of the normal at P is $\dfrac{a\cos t}{b\sin t}$

Knowing that the coordinates of the foci are $(\pm ae, 0)$, we can easily find that the gradients of PS and PS' are $\dfrac{b\sin t}{a\cos t \pm ae}$

So, using the result from part **a**, the tangent of the angle between PS and the normal is:

$$\frac{\dfrac{a\sin t}{b\cos t} - \dfrac{b\sin t}{a\cos t - ae}}{1+\left(\dfrac{a\sin t}{b\cos t}\right)\left(\dfrac{b\sin t}{a\cos t - ae}\right)} = \frac{\dfrac{a^2\cos t\sin t - a^2e\sin t - b^2\cos t\sin t}{ab\cos^2 t - abe\cos t}}{1+\dfrac{ab\sin^2 t}{ab\cos^2 t - abe\cos t}}$$

$$= \frac{(a^2-b^2)\cos t\sin t - a^2e\sin t}{ab\cos^2 t - abe\cos t + ab\sin^2 t}$$

$$= \frac{a^2e^2\cos t\sin t - a^2e\sin t}{ab(1-e\cos t)}$$

$$= \frac{-a^2e\sin t(1-e\cos t)}{ab(1-e\cos t)} = -\frac{ae\sin t}{b}$$

Similarly, we find that this is also the value of the tangent of the angle between the normal and PS'
Therefore, since the tangent is injective between 0 and 2π (where it is defined), we can conclude that the two angles are the same.

2 **a** $y = \frac{1}{x},\ x > 1$

$$V = \pi\int_a^b y^2\,\mathrm{d}x$$

$$= \pi\int_1^\infty \left(\frac{1}{x^2}\right)\mathrm{d}x$$

$$= \pi\left[-\frac{1}{x}\right]_1^\infty$$

$$= \pi\left(-\frac{1}{\infty}+\frac{1}{1}\right)$$

$$= \pi$$

b $y = \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

Using $S = 2\pi\int_{x_A}^{x_B} y\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x$ gives:

$$S = 2\pi\int_1^\infty \frac{1}{x}\left(1+\left(-\frac{1}{x^2}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x$$

$S = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\mathrm{d}x$ as required.

c $\sqrt{1+\frac{1}{x^4}} > 1,\ x > 0$ is always positive, therefore:

$$\frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \frac{1}{x}$$

and

$2\pi\int_1^a \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\mathrm{d}x > 2\pi\int_1^a \frac{1}{x}\mathrm{d}x,\ x > 0$ as required.

d $2\pi\int_1^a \frac{1}{x}\mathrm{d}x = 2\pi\left[\ln x\right]_1^a$

$$= 2\pi\ln a$$

As $a \to \infty$, $\ln a \to \infty$

Therefore:

$2\pi\int_1^a \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\ \mathrm{d}x \to \infty$ as $a \to \infty$

So Torricelli's trumpet does have infinite surface area.